Lecture 2c - More on Boltzmann Eqn.

Recap:

- derived Boltzmann Equation from BBGKY Hierarchy
  - ⇒ $nd^3 \ll 1$ to truncate at $f(1,2)$ eqn.
  - ⇒ $f(1,2) = f(1)\,f(2)$ to complete

- $C(f) = \int d\underline{p}_1 \int d\underline{p}' \int d\underline{p}_1'\; w(\underline{p},\underline{p}_1;\underline{p}',\underline{p}_1') *$
$$\big(f(\underline{p}')\,f(\underline{p}_1') - f(\underline{p}_1)\,f(\underline{p})\big)$$

- Proved H-Theorem, using Lemma and Stosszahlansatz

$$S = -\int d\underline{x} \int d\underline{p}\; f \ln f$$

$$\frac{dS}{dt} = -\int d\underline{x} \int d\underline{p} \; C(f) \ln f$$

$$\frac{dS}{dt} \geq 0.$$

Macroscopic irreversibility from microscopically reversible dynamics. $w = w^T$

- When is $\frac{dS}{dt} = 0$ $\Rightarrow$ f s/t

$C(f) = 0$?

$\Rightarrow$ identifies equilibrium distribution

Recall: $\frac{dS}{dt} = \frac{1}{2} \int d\underline{x} \int d\underline{p} \; w f f_1 \; x \ln x$

$x = f' f_1' / f f_1$

so $\frac{dS}{dt} = 0$ at $x = 1$

$$f f_1 = f' f_1'$$

$$\ln f + \ln f_1 = \ln f' + \ln f_1'$$

so $\ln f + \ln f_1$ must be conserved in collision event

$\Rightarrow$

$$\ln f = c + \underset{\text{momentum}}{\underline{\alpha} \cdot \underline{p}} + \beta \underset{\text{energy}}{\epsilon}$$

Now, f normalizable; $\quad \epsilon = \frac{p^2}{2m}$

$$\ln f = c + \underline{\alpha} \cdot \underline{p} - \beta \epsilon$$

$\beta > 0$

And $\int d\underline{x}$ irrelevant, i.e. can specify each $\underline{x}$ $\quad C(f_{eq}) = 0 \Rightarrow f_{eq} = f_{eq}(\underline{x})$,

$c = c(\underline{x}) \rightarrow n(\underline{x})$
$\underline{\alpha} = \alpha(\underline{x}) \rightarrow \underline{V}(\underline{x})$
$\beta = \beta(\underline{x}) \rightarrow 1/T(\underline{x})$

normalization

$$f_{eq} = c\, n(\underline{x}) \exp\left[(\underline{p}\cdot \underline{V}(x) - \epsilon)/T(x)\right]$$

i.e. shifted Maxwellian, as expected!

- Local vs. Global?

→ $\int d\underline{x}$ irrelevant

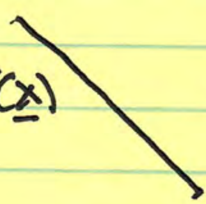
i.e. $C(f_{eq}) = 0$ at each $\underline{x}$

Entropy produced locally, on

ν time scale $(\omega \sim \sigma V_{rel})$

$(\nu = v_{th}/l_{mfp} \quad l_{mfp} = 1/n\sigma)$

→ But: What of inhomogeneity in thermodynamic parameters?

$T(\underline{x})$  $\Rightarrow \underline{Q}$ $\Rightarrow$ $T(\underline{x})$ 

The point:

$f_{eq} = f_{eq}(T(\underline{x}), V(\underline{x}), n(\underline{x}))$ does <u>not</u> satisfy Boltzmann Equation,

$$\partial_t f + \underline{v}\cdot\underline{\nabla} f = C(f)$$

$$\cancel{\partial_t f_{eq}} + \underline{v}\cdot\underline{\nabla} f_{eq} = \cancel{C(f_{eq})} \rightarrow 0$$

$$\underline{v}\cdot\nabla \underline{f}_{eq} \neq 0$$

!

<u>so</u>

$$f \rightarrow f_{eq} + \delta f \quad \rightarrow O(l_{mfp}/L)$$

Rate: $\sim \left(\frac{l_{mfp}}{L}\right)^2$ induces collisional fluxes, etc.

- How reconcile:
  - reversible Hamiltonian dynamics
  - $dS/dt \geq 0$ <u>Macro</u> irreversibility

→ Coarse Graining ! ↔ Partition

(recall Lyapunov Exponents

→ partition

$\Delta q \Delta p$ – partition cell
sets resolution scale

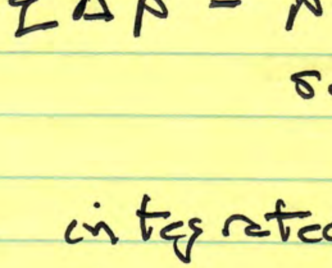
S is integrated quantity

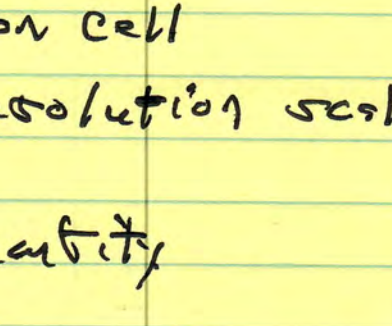
Why significant?

→ partition kills small details in phase volume evolution.

$t=0$ exact

i.e

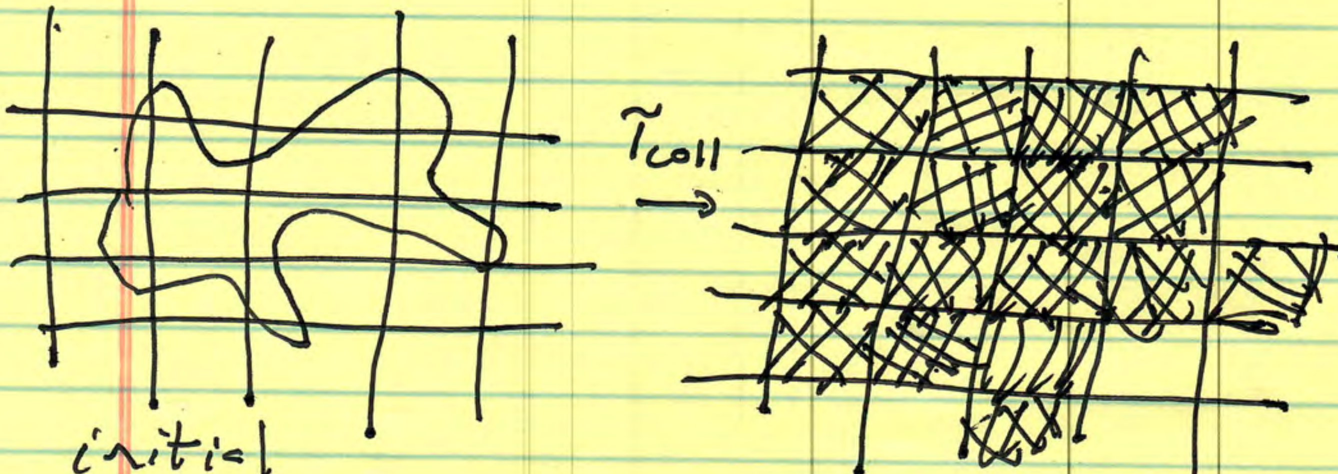




via Hamiltonian evolution

2 time scales — step — Hamiltonian Mapping
— τ_c

with coarse graining:



coarse grained area

{local phase space density modified by coarse graining

so

$$f_0 A_0 = A_{CG} \bar{f} \qquad A_0 < A_{CG}$$

$$\bar{f} = f_0 A_0 / A_{CG} \qquad \bar{f} < f_0$$

↓
coarse grained phase space density

N.B. Prediction of close recurrence impossible as partition sets resolution limit

What Next ?

→ We have the Boltzmann Equation and H-Theorem. (Yay!)

→ What do we do with them?

N.B.: "The solution of the above equation [B.E.], as we will see shortly, is truly a gruesome task"

- Stewart Harris, in "An Introduction to the Theory of the Boltzmann Equation".

So ?

Hint: Consider time scales ...